

# Hunt's hypothesis (H) and Gettoor's conjecture for Lévy processes

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**Abstract** In this paper, Hunt's hypothesis (H) and Gettoor's conjecture for Lévy processes are revisited. Let  $X$  be a Lévy process on  $\mathbf{R}^n$  with Lévy-Khintchine exponent  $(a, A, \mu)$ . First, we show that if  $A$  is non-degenerate then  $X$  satisfies (H). Second, under the assumption that  $\mu(\mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n) < \infty$ , we show that  $X$  satisfies (H) if and only if the equation

$$\sqrt{A}y = -a - \int_{\{x \in \mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n : |x| < 1\}} x \mu(dx), \quad y \in \mathbf{R}^n,$$

has at least one solution. Finally, we show that if  $X$  is a subordinator and satisfies (H) then its drift coefficient must be 0.

**Keywords** Hunt's hypothesis, Gettoor's conjecture, Lévy processes

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## 1 Introduction and main results

Let  $X$  be a nice Markov process. Hunt's hypothesis (H) says that “every semipolar set of  $X$  is polar”. (H) plays a crucial role in the potential theory of (dual) Markov processes. We refer the reader to Blumenthal and Gettoor [1, Chapter VI], [2] for details. In spite of its importance, (H) has been verified only in some special situations. Let  $X$  and  $\hat{X}$  be a pair of dual Markov processes as in [1, Chapter VI]. Then, (H) holds if and only if the fine and cofine topologies differ by polar sets, see [1, VI.4.10] and Glover [8, Theorem (2.2)]. Some forty years ago, Gettoor conjectured that essentially all Lévy processes satisfy (H).

Throughout this paper, we let  $(\Omega, \mathcal{F}, P)$  be a probability space and  $X = (X_t)_{t \geq 0}$  be a  $\mathbf{R}^n$ -valued Lévy process on  $(\Omega, \mathcal{F}, P)$  with Lévy-Khintchine exponent  $\psi$ , i.e.,

$$E[\exp\{i\langle z, X_t \rangle\}] = \exp\{-t\psi(z)\}, \quad z \in \mathbf{R}^n, t \geq 0,$$

where  $E$  denotes the expectation with respect to  $P$ . For  $\psi$ , we have the following famous Lévy-Khintchine formula:

$$\psi(z) = i\langle a, z \rangle + \frac{1}{2}\langle z, Az \rangle + \int_{\mathbf{R}^n} (1 - e^{i\langle z, x \rangle} + i\langle z, x \rangle 1_{\{|x| < 1\}}) \mu(dx),$$

where  $a \in \mathbf{R}^n$ ,  $A$  is a symmetric nonnegative definite  $n \times n$  matrix, and  $\mu$  is a measure (called the Lévy measure) on  $\mathbf{R}^n \setminus \{0\}$  satisfying  $\int_{\mathbf{R}^n \setminus \{0\}} (1 \wedge |x|^2) \mu(dx) < \infty$ . Hereafter, we use  $\operatorname{Re}(\psi)$  and  $\operatorname{Im}(\psi)$  to denote the real part and imaginary part of  $\psi$ , respectively, and use  $(a, A, \mu)$  to denote  $\psi$  sometimes. For every  $x \in \mathbf{R}^n$ , we denote by  $P^x$  the law of  $x + X$  under  $P$ . In particular,  $P^0 = P$ .

Let  $B \subset \mathbf{R}^n$ . We define the first hitting time of  $B$  by

$$\sigma_B := \inf\{t > 0 : X_t \in B\}.$$

Denote by  $\mathcal{B}^*$  the family of all nearly Borel sets relative to  $X$  (cf. [1, I.10.21]). A set  $B \subset \mathbf{R}^n$  is called polar (resp. essentially polar) if there exists a set  $C \in \mathcal{B}^*$  such that  $B \subset C$  and  $P^x(\sigma_C < \infty) = 0$  for every  $x \in \mathbf{R}^n$  (resp.  $dx$ -almost every  $x \in \mathbf{R}^n$ ). Hereafter  $dx$  denotes the Lebesgue measure on  $\mathbf{R}^n$ .  $B$  is called a thin set if there exists a set  $C \in \mathcal{B}^*$  such that  $B \subset C$  and  $P^x(\sigma_C = 0) = 0$  for every  $x \in \mathbf{R}^n$ .  $B$  is called semipolar if  $B \subset \cup_{n=1}^{\infty} B_n$  for some thin sets  $\{B_n\}_{n=1}^{\infty}$ .

Before introducing our results, we first recall some important results obtained so far for Gettoor's conjecture. When  $n = 1$ , Kesten [15] (cf. also Bretagnolle [3]) showed that if  $X$  is not a compound Poisson process, then every  $\{x\}$  is non-polar if and only if

$$\int_0^{\infty} \operatorname{Re}([1 + \psi(z)]^{-1}) dz < \infty.$$

(If  $X$  is a compound Poisson process, then it is easy to see that every  $x$  is regular for  $\{x\}$ , i.e.,  $P^x(\sigma_{\{x\}} > 0) = 0$ .) Port and Stone [17] proved that for the asymmetric Cauchy process on the line every  $x$  is regular for  $\{x\}$ . Hence only the empty set is a semipolar set and therefore (H) holds in this case. Further, Blumenthal and Gettoor [2] showed that all stable processes with index  $\alpha \in (0, 2)$  on the line satisfy (H).

Kanda [13] and Forst [5] proved that (H) holds if  $X$  has bounded continuous transition densities (with respect to  $dx$ ) and the Lévy-Khintchine exponent  $\psi$  satisfies  $|\operatorname{Im}(\psi)| \leq M(1 + \operatorname{Re}(\psi))$  for some positive constant  $M$ . Rao [18] gave a short proof of the Kanda-Forst theorem under the weaker condition that  $X$  has resolvent densities. In particular, for  $n > 1$  all stable processes of index  $\alpha \neq 1$  satisfy (H). Kanda [14] settled this problem for the case  $\alpha = 1$  assuming the linear term vanishes. Silverstein [20] extended the Kanda-Forst condition to the non-symmetric Dirichlet forms setting, and Fitzsimmons [4] extended it to the semi-Dirichlet forms setting. Glover and Rao [9] proved that  $\alpha$ -subordinates of general Hunt processes satisfy (H). Rao [19] proved that if all 1-excessive functions of  $X$  are lower semicontinuous and  $|\operatorname{Im}(\psi)| \leq (1 + \operatorname{Re}(\psi))f(1 + \operatorname{Re}(\psi))$ , where  $f$  is an increasing function on  $[1, \infty)$  such that  $\int_N^{\infty} (zf(z))^{-1} dz = \infty$  for any  $N \geq 1$ , then  $X$  satisfies (H).

Now we introduce the main results of this paper. To state the first result, we let  $\bar{X}$  be an independent copy of  $X$ . Define the symmetrization  $\tilde{X}$  of  $X$  by  $\tilde{X} := X - \bar{X}$ .

**Theorem 1.1** *Suppose that  $A$  is non-degenerate, i.e.,  $A$  is of full rank. Then:*

- (i)  $X$  satisfies (H);
- (ii) The Kanda-Forst condition  $|\operatorname{Im}(\psi)| \leq M(1 + \operatorname{Re}(\psi))$  holds for some positive constant  $M$ ;
- (iii)  $X$  and  $\tilde{X}$  have the same polar sets.

Denote  $b := -a$  and  $\mu_1 := \mu|_{\mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n}$ . If  $\int_{|x|<1} |x| \mu_1(dx) < \infty$ , we set  $b' := b - \int_{|x|<1} x \mu_1(dx)$ . To state the second result, we define the following solution condition:

(S) The equation  $\sqrt{A}y = b'$ ,  $y \in \mathbf{R}^n$ , has at least one solution.

**Theorem 1.2** *Suppose that  $\mu(\mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n) < \infty$ . Then, the following three claims are equivalent:*

- (i)  $X$  satisfies (H);
- (ii) (S) holds;
- (iii) The Kanda-Forst condition  $|\operatorname{Im}(\psi)| \leq M(1 + \operatorname{Re}(\psi))$  holds for some positive constant  $M$ .

**Remark 1.3** (i) *Theorem 1.1 tells us that if a Lévy process on  $\mathbf{R}^n$  is perturbed by an independent (small)  $n$ -dimensional Brownian motion, then the perturbed Lévy process must satisfy (H).*

(ii) *By Theorem 1.2 and Jacob [12, Example 4.7.32], one finds that if  $X$  satisfies (H) and  $\mu(\mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n) < \infty$ , then  $X$  must be associated with a Dirichlet form on  $L^2(\mathbf{R}^n; dx)$ .*

**Proposition 1.4** *Suppose that  $\mu(\mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n) < \infty$ . Then:*

- (i)  $X$  has transition densities implies that all the claims of Theorem 1.2 are fulfilled.
- (ii) If one of the claims in Theorem 1.2 is fulfilled, then the following four claims are equivalent:
  - (a) Every essentially polar set of  $X$  is polar;
  - (b)  $X$  has resolvent densities;
  - (c)  $X$  has transition densities;
  - (d)  $A$  is of full rank.

**Proposition 1.5** *Suppose that  $X$  has bounded continuous transition densities, and  $X$  and  $\tilde{X}$  have the same polar sets. Then  $X$  satisfies (H).*

Suppose that  $X$  is a subordinator. Then  $\psi$  can be expressed by

$$\psi(z) = -idz + \int_{(0,\infty)} (1 - e^{izx}) \mu(dx), \quad z \in \mathbf{R},$$

where  $d \geq 0$  (called the drift coefficient) and  $\mu$  satisfies  $\int_{(0,\infty)} (1 \wedge x) \mu(dx) < \infty$ .

**Proposition 1.6** *If  $X$  is a subordinator and satisfies (H), then  $d = 0$ .*

The rest of this paper is organized as follows. In Section 2, we recall the Lévy-Itô decomposition of Lévy processes and discuss the orthogonal transformation of Lévy processes. In Section 3, we present the proofs of our results.

## 2 Lévy-Itô decomposition and orthogonal transformation of Lévy processes

### 2.1 Lévy-Itô decomposition

**Theorem 2.1 (Lévy-Itô)** *Let  $X$  be a Lévy process on  $\mathbf{R}^n$  with exponent  $(a, A, \mu)$ . Then there exist a Brownian motion  $B_A$  on  $\mathbf{R}^n$  with covariance matrix  $A$  and an independent Poisson random measure  $N$  on  $\mathbf{R}^+ \times (\mathbf{R}^n \setminus \{0\})$  such that, for each  $t \geq 0$ ,*

$$X_t = bt + B_A(t) + \int_{|x| \geq 1} xN(t, dx) + \int_{|x| < 1} x\tilde{N}(t, dx), \quad (2.1)$$

where  $b = -a$ ,  $\tilde{N}(t, F) = N(t, F) - t\mu(F)$ .

Define

$$X_t^{(I)} := bt + B_A(t), \quad X_t^{(II)} := \int_{|x| \geq 1} xN(t, dx), \quad X_t^{(III)} := \int_{|x| < 1} x\tilde{N}(t, dx), \quad t \geq 0.$$

Then  $X^{(I)}$ ,  $X^{(II)}$  and  $X^{(III)}$  are mutually independent,  $X^{(II)}$  is a compound Poisson process, and  $X^{(III)}$  is a square integrable martingale. For convenience, we write  $X_t^{(I)} = bt + \sqrt{A}B_t$ , where  $B = (B_t)_{t \geq 0}$  is a standard  $n$ -dimensional Brownian motion.

### 2.2 Orthogonal transformation

Let  $X$  be a Lévy process on  $\mathbf{R}^n$  with exponent  $(a, A, \mu)$ . Since  $A$  is a symmetric nonnegative definite matrix, there exists an orthogonal matrix  $O$  such that

$$OAO^T = \text{diag}(\lambda_1, \dots, \lambda_n) := D,$$

where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$  and  $O^T$  denotes the transpose of  $O$ . We fix such an orthogonal matrix  $O$  and define  $Y_t := OX_t$ ,  $t \geq 0$ . Then  $Y = (Y_t)_{t \geq 0}$  is a Lévy process on  $\mathbf{R}^n$  and  $X$  satisfies (H) if and only if  $Y$  satisfies (H). We will see that sometimes it is more convenient to work with  $Y$ . By the expression of the exponent of  $X$  and simple computation, we get that the exponent of  $Y$  is  $(Oa, D, \mu O^{-1})$ , where  $\mu O^{-1}(B) = \mu(\{x \in \mathbf{R}^n : Ox \in B\})$  for any Borel set  $B$  of  $\mathbf{R}^n$ .

From now on, we denote by  $k$  the rank of  $A$ . Then, the orthogonal transformation satisfies the following properties:

(1)  $\mu(\mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n) < \infty$  if and only if  $\mu O^{-1}$  is a finite measure on  $\mathbf{R}^k \times (\mathbf{R}^{n-k} \setminus \{0\})$ . (When  $k = n$ ,  $\mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n$  and  $\mathbf{R}^k \times (\mathbf{R}^{n-k} \setminus \{0\})$  are the empty set.)

(2) If  $\int_{|x|<1} |x| \mu_1(dx) < \infty$ , then

$$\int_{\{y \in \mathbf{R}^k \times (\mathbf{R}^{n-k} \setminus \{0\}) : |y| < 1\}} |y| \mu O^{-1}(dy) = \int_{|y| < 1} |y| \mu_1 O^{-1}(dy) = \int_{|x| < 1} |x| \mu_1(dx) < \infty.$$

Recall that  $b' = b - \int_{|x|<1} x \mu_1(dx)$ . Define  $\bar{b} := Ob'$ . Then

$$\bar{b} = Ob - \int_{\{y \in \mathbf{R}^k \times (\mathbf{R}^{n-k} \setminus \{0\}) : |y| < 1\}} y \mu O^{-1}(dy).$$

Note that  $\sqrt{A} = O^T \sqrt{D} O$ . Then, the equation  $\sqrt{A}y = b'$  is equivalent to  $\sqrt{D}Oy = Ob'$ . Therefore, the equation  $\sqrt{A}y = b', y \in \mathbf{R}^n$ , has a solution if and only if the equation  $\sqrt{D}y = \bar{b}, y \in \mathbf{R}^n$ , has a solution.

(3) Suppose that  $\int_{|x|<1} |x| \mu_1(dx) < \infty$ . Then, by the Lévy-Itô decomposition (2.1),  $Y$  can be expressed by

$$Y_t = Ob t + \sqrt{D} \bar{B}_t + \int_{|y| \geq 1} y \bar{N}(t, dy) + \int_{|y| < 1} y \tilde{N}(t, dy), \quad (2.2)$$

where  $\bar{B} = OB$  is a standard Brownian motion on  $\mathbf{R}^n$ ,  $\bar{N}$  is a Poisson random measure on  $\mathbf{R}^+ \times (\mathbf{R}^n \setminus \{0\})$  with  $\mu O^{-1}$  being its intensity measure,  $\tilde{N}(t, F) = \bar{N}(t, F) - t \mu O^{-1}(F)$ ,  $\bar{B}$  and  $\bar{N}$  are independent. We rewrite (2.2) as

$$Y_t = Y_t^{(1)} + Y_t^{(2)},$$

where

$$\begin{aligned} Y_t^{(1)} &:= \bar{b} t + \sqrt{D} \bar{B}_t + \int_{\{y \in \mathbf{R}^k \times \{0\} : |y| \geq 1\}} y \bar{N}(t, dy) + \int_{\{y \in \mathbf{R}^k \times \{0\} : |y| < 1\}} y \tilde{N}(t, dy), \\ Y_t^{(2)} &:= \int_{\mathbf{R}^k \times (\mathbf{R}^{n-k} \setminus \{0\})} y \bar{N}(t, dy), \end{aligned}$$

$Y^{(1)}$  and  $Y^{(2)}$  are independent.

By (2), we can see that (S) holds if and only if  $\bar{b} \in \mathbf{R}^k \times \{0\}$ . In this case,  $Y^{(1)}$  can be regarded as a  $k$ -dimensional Lévy process on  $\mathbf{R}^k \times \{0\}$ , which has a non-degenerate Gaussian component. If  $\mu(\mathbf{R}^n \setminus \sqrt{A}\mathbf{R}^n) < \infty$ , then  $Y^{(2)}$  is a compound Poisson process.

### 3 Proofs of the main results

#### 3.1 Proof of Theorem 1.1

First, we prove (ii). Since  $A$  is of full rank, there exists a constant  $c > 0$  such that  $\langle z, Az \rangle \geq c\langle z, z \rangle$ ,  $\forall z \in \mathbf{R}^n$ . Then

$$\operatorname{Re}\psi(z) = \frac{1}{2}\langle z, Az \rangle + \int_{\mathbf{R}^n} (1 - \cos\langle z, x \rangle) \mu(dx) \geq \frac{1}{2}\langle z, Az \rangle \geq \frac{c}{2}\langle z, z \rangle. \quad (3.1)$$

By the Cauchy-Schwarz inequality, one finds that  $|\langle a, z \rangle|$  is controlled by  $1 + \operatorname{Re}\psi(z)$ . To establish the Kanda-Forst condition, we need only show that  $|\operatorname{Im}\{\int_{|x|<1} (1 - e^{i\langle z, x \rangle} + i\langle z, x \rangle) \mu(dx)\}|$  is controlled by  $\langle z, z \rangle$ . Note that  $|t - \sin t| \leq t^2/2$  for any  $t \in \mathbf{R}$ . Then,

$$\begin{aligned} \left| \operatorname{Im} \left\{ \int_{|x|<1} (1 - e^{i\langle z, x \rangle} + i\langle z, x \rangle) \mu(dx) \right\} \right| &= \left| \int_{|x|<1} (\langle z, x \rangle - \sin\langle z, x \rangle) \mu(dx) \right| \\ &\leq \frac{1}{2} \int_{|x|<1} |\langle z, x \rangle|^2 \mu(dx) \\ &\leq \left( \frac{1}{2} \int_{|x|<1} |x|^2 \mu(dx) \right) |z|^2. \end{aligned}$$

Therefore (ii) holds.

Second, we prove (i). By (3.1), we get

$$\lim_{|z| \rightarrow \infty} \frac{\operatorname{Re}\psi(z)}{\ln(1 + |z|)} = \infty. \quad (3.2)$$

By Hartman and Wintner [10] (cf. also Knopova and Schilling [16]) and (3.2), we find that  $X$  has bounded continuous transition densities. Then, by (ii) and the Kanda-Forst theorem, we obtain (i).

Finally, we prove (iii). Denote by  $\tilde{\psi}$  the Lévy-Khintchine exponent of  $\tilde{X}$ . Note that  $\tilde{\psi} = 2\operatorname{Re}(\psi)$ . Then, for any  $\lambda \geq 1$ , by (ii) we get (cf. Kanda [13, Page 163])

$$\begin{aligned} 2\operatorname{Re} \left( \frac{1}{\lambda + \tilde{\psi}(\xi)} \right) &= \frac{1}{\frac{1}{2}\lambda + \frac{1}{2}\tilde{\psi}(\xi)} \geq \frac{1}{\lambda + \operatorname{Re}\psi(\xi)} \\ &\geq \operatorname{Re} \left( \frac{1}{\lambda + \psi(\xi)} \right) \\ &= \frac{\lambda + \operatorname{Re}\psi(\xi)}{(\lambda + \operatorname{Re}\psi(\xi))^2 + (\operatorname{Im}\psi(\xi))^2} \\ &= \frac{1}{\lambda + \operatorname{Re}\psi(\xi)} \left[ 1 + \left( \frac{\operatorname{Im}\psi(\xi)}{\lambda + \operatorname{Re}\psi(\xi)} \right)^2 \right]^{-1} \\ &\geq \frac{1}{\lambda + \operatorname{Re}\psi(\xi)} \left[ 1 + \left( \frac{M(1 + \operatorname{Re}\psi(\xi))}{\lambda + \operatorname{Re}\psi(\xi)} \right)^2 \right]^{-1} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{(1+M^2)(\lambda + \frac{1}{2}\tilde{\psi}(\xi))} \\
&\geq \frac{1}{1+M^2} \operatorname{Re} \left( \frac{1}{\lambda + \tilde{\psi}(\xi)} \right). \tag{3.3}
\end{aligned}$$

By (3.3), the above proved fact that  $X$  has bounded continuous transition densities, and Kanda [13, Theorem 1] (or Hawkes [11, Theorems 2.1 and 3.3]), we obtain (iii).  $\square$

### 3.2 Proof of Theorem 1.2

By the discussion of §2.2, we know that  $X$  satisfies (H) if and only if  $Y$  satisfies (H), and (S) holds if and only if  $\bar{b} \in \mathbf{R}^k \times \{0\}$ . By the expression of the exponent of  $Y$ , it is easy to see that the Kanda-Forst condition holds for  $X$  if and only if it holds for  $Y$ . Hence, to prove Theorem 1.2, we may and do assume without loss of generality that  $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n) := D$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0, \lambda_{k+1} = \lambda_{k+2} = \dots = \lambda_n = 0$  ( $k \geq 0$ ), and  $X$  has the expression

$$X_t = X_t^{(1)} + X_t^{(2)}, \quad t \geq 0,$$

where

$$\begin{aligned}
X_t^{(1)} &:= b't + \sqrt{D}B_t + \int_{\{x \in \mathbf{R}^k \times \{0\} : |x| \geq 1\}} xN(t, dx) + \int_{\{x \in \mathbf{R}^k \times \{0\} : |x| < 1\}} x\tilde{N}(t, dx), \tag{3.4} \\
X_t^{(2)} &:= \int_{\mathbf{R}^k \times (\mathbf{R}^{n-k} \setminus \{0\})} xN(t, dx),
\end{aligned}$$

$b'$  is the same as in §1, and  $B$ ,  $N$  and  $\tilde{N}$  are the same as in §2.1.

If  $k = 0$ , then  $X_t = b't + X_t^{(2)}$ . Since  $X^{(2)}$  is a compound Poisson process, it is easy to see that (i), (ii) and (iii) are equivalent in this case. Below we assume that  $k \geq 1$ .

(ii)  $\Rightarrow$  (iii): Suppose that (S) holds, i.e.,  $b' \in \mathbf{R}^k \times \{0\}$ . Then  $X^{(1)}$  stays in  $\mathbf{R}^k \times \{0\}$  if it starts there. By Theorem 1.1, the Kanda-Forst condition holds for  $X^{(1)}$ . Since  $X^{(2)}$  is a compound Poisson process, its Lévy-Khintchine exponent is bounded. Hence the Kanda-Forst condition holds for  $X$ , i.e., (iii) holds.

(iii)  $\Rightarrow$  (ii): Suppose that the Kanda-Forst condition holds for  $X$ . Since the Lévy-Khintchine exponent of  $X^{(2)}$  is bounded, we get that the Kanda-Forst condition holds for  $X^{(1)}$ . Assume that  $b' \notin \mathbf{R}^k \times \{0\}$ . We will reach a contradiction. Denote  $b' = (b'_1, \dots, b'_n)$ . Without loss of generality, we assume that  $b'_n \neq 0$ . Let  $\psi_1$  be the Lévy-Khintchine exponent of  $X^{(1)}$ . Then

$$\psi_1(z) = i\langle b', z \rangle + \frac{1}{2}\langle z, \sqrt{D}z \rangle + \int_{\mathbf{R}^k \times \{0\}} (1 - e^{i\langle z, x \rangle} + i\langle z, x \rangle 1_{\{|x| < 1\}}) \mu(dx).$$

It follows that if  $z = (z_1, \dots, z_n)$  with  $z_i = 0, i = 1, \dots, n-1$  and  $z_n \neq 0$ , then  $\psi_1(z) = b'_n z_n i$  and thus the Kanda-Forst condition cannot hold for  $X^{(1)}$ . Hence  $b' \in \mathbf{R}^k \times \{0\}$  and therefore (S) holds.

(i)  $\Rightarrow$  (ii): We will show  $b' \notin \mathbf{R}^k \times \{0\}$  implies that  $X$  does not satisfy (H). We first consider the case that  $\mu_1 \neq 0$ .

Suppose that  $b' \notin \mathbf{R}^k \times \{0\}$ . First, we show that  $\mathbf{R}^k \times \{0\}$  is a thin set of  $X$ . Let  $T_1^{(2)}$  be the first jumping time of  $X^{(2)}$ . Since  $X^{(2)}$  is a compound Poisson process,  $T_1^{(2)}$  has an exponential distribution, in particular,

$$P(T_1^{(2)} > 0) = 1. \quad (3.5)$$

For any  $x \in \mathbf{R}^k \times \{0\}$  and any  $t > 0$ , we know that  $x + X_t^{(1)} \notin \mathbf{R}^k \times \{0\}$  since  $b' \notin \mathbf{R}^k \times \{0\}$ , which together with (3.5) implies that

$$\begin{aligned} P^x(\sigma_{\mathbf{R}^k \times \{0\}} = 0) &\leq P^0\left(\exists t \in (0, T_1^{(2)}) \text{ s.t. } x + X_t \in \mathbf{R}^k \times \{0\}\right) \\ &= P^0\left(\exists t \in (0, T_1^{(2)}) \text{ s.t. } x + X_t^{(1)} \in \mathbf{R}^k \times \{0\}\right) \\ &= 0. \end{aligned} \quad (3.6)$$

For any  $x \notin \mathbf{R}^k \times \{0\}$ , the distance between  $x$  and the subspace  $\mathbf{R}^k \times \{0\}$  is strictly positive. By (3.5) and the right continuity of the sample path of  $X^{(1)}$ , we get

$$\begin{aligned} P^x(\sigma_{\mathbf{R}^k \times \{0\}} = 0) &= P^0\left(\exists \{t_n, n \geq 1\} \subset (0, T_1^{(2)}) \text{ s.t. } x + X_{t_n} \in \mathbf{R}^k \times \{0\}, t_n \downarrow 0\right) \\ &= P^0\left(\exists \{t_n, n \geq 1\} \subset (0, T_1^{(2)}) \text{ s.t. } x + X_{t_n}^{(1)} \in \mathbf{R}^k \times \{0\}, t_n \downarrow 0\right) \\ &= 0. \end{aligned} \quad (3.7)$$

It follows from (3.6) and (3.7) that  $\mathbf{R}^k \times \{0\}$  is a thin set and thus a semipolar set of  $X$ .

Next, we show that  $\mathbf{R}^k \times \{0\}$  is not a polar set of  $X$ . Note that  $P^0(T_1^{(2)} > s) > 0$  for any  $s > 0$ . Then

$$\begin{aligned} P^{-b's}(\sigma_{\mathbf{R}^k \times \{0\}} < \infty) &= P^{-b's}(\exists t > 0 \text{ s.t. } X_t \in \mathbf{R}^k \times \{0\}) \\ &\geq P^{-b's}(X_s \in \mathbf{R}^k \times \{0\}) \\ &= P^0((X_s - b's) \in \mathbf{R}^k \times \{0\}) \\ &\geq P^0(T_1^{(2)} > s) \\ &> 0. \end{aligned}$$

Hence  $\mathbf{R}^k \times \{0\}$  is not a polar set of  $X$ . Therefore  $X$  does not satisfy (H).

The case that  $\mu_1 = 0$  can be proved similarly by  $T_1^{(2)} \equiv \infty$ .

(ii)  $\Rightarrow$  (i): Suppose that (S) holds, i.e.,  $b' \in \mathbf{R}^k \times \{0\}$ . Let  $F$  be a semipolar set of  $X$ . We will show that  $F$  is a polar set of  $X$ . Without loss of generality, we assume that  $F$  is a nearly Borel set. For  $y \in \mathbf{R}^{n-k}$ , we define

$$F_y := F \cap (\mathbf{R}^k \times \{y\}).$$



Since  $X^{(2)}$  is a compound Poisson process, one finds that  $F_y$  is semipolar for the  $k$ -dimensional Lévy process  $(X^{(1)}, P^{(x,y)})_{x \in \mathbf{R}^k}$  on  $\mathbf{R}^k \times \{y\}$ . Hence  $F_y$  is polar for  $(X^{(1)}, P^{(x,y)})_{x \in \mathbf{R}^k}$  by Theorem 1.1. Therefore,

$$P\left(\exists t > 0 \text{ s.t. } (x, y) + X_t^{(1)} \in F_y\right) = 0, \quad \forall x \in \mathbf{R}^k, \forall y \in \mathbf{R}^{n-k}. \quad (3.8)$$

Denote by  $\eta$  the distribution of  $T_1^{(2)}$  under  $P$ . Let  $\xi$  be a random variable taking values on  $\mathbf{R}^k \times (\mathbf{R}^{n-k} \setminus \{0\})$ , which has distribution  $\mu_1$  and is independent of  $X^{(1)}$  and  $T_1^{(2)}$ . Then, for any  $x_0 = (u, v) \in \mathbf{R}^k \times \mathbf{R}^{n-k}$ , we obtain by (3.8) that

$$\begin{aligned} P\left(x_0 + X_{T_1^{(2)}}^{(1)} + \xi \in F\right) &= \int_{\mathbf{R}^k \times (\mathbf{R}^{n-k} \setminus \{0\})} \int_{(0, \infty)} P((u, v) + X_t^{(1)} + (x, y) \in F) \eta(dt) \mu_1(dx, dy) \\ &= \int_{\mathbf{R}^k \times (\mathbf{R}^{n-k} \setminus \{0\})} \int_{(0, \infty)} P((u + x, v + y) + X_t^{(1)} \in F_{v+y}) \eta(dt) \mu_1(dx, dy) \\ &= 0. \end{aligned}$$

Since  $x_0$  is arbitrary, by the strong Markov property of Lévy process,  $F$  is a polar set of  $X$ . Therefore,  $X$  satisfies (H).  $\square$

### 3.3 Proof of Proposition 1.4

(i) Suppose that  $X$  has transition densities. We will show that  $A$  is of full rank. We adopt the setting of §3.2. Assume that  $k < n$ . Set  $X = (X^1, \dots, X^n)$  and  $b' = (b^1, \dots, b^n)$ . Without loss of generality, we suppose  $\mu_1 \neq 0$ . Let  $T_1^{(2)}$  be the first jumping time of  $X^{(2)}$ . Then  $T_1^{(2)}$  has an exponential distribution and thus  $P(T_1^{(2)} > 1) > 0$ . It follows from (3.4) that  $P(X_1^n = b^n) > 0$ . This contradicts with the assumption that  $X$  has transition densities. Hence  $A$  is of full rank. Therefore, the proof is completed by Theorem 1.1.

(ii) (a)  $\Leftrightarrow$  (b) follows from Fukushima [6, (viii)]. (d)  $\Rightarrow$  (c)  $\Rightarrow$  (b) is easy. (b)  $\Rightarrow$  (c) follows from the Kanda-Forst condition, Silverstein [21, Theorem 3.2] and the spatial homogeneity of Lévy processes. (Note that if the Lévy process  $X$  is associated with a Dirichlet form on  $L^2(\mathbf{R}^n; dx)$ , then the Dirichlet form is regular by Silverstein [20, Lemma 1.5].) (c)  $\Rightarrow$  (d) follows from the above proof of (i).  $\square$

### 3.4 Proof of Proposition 1.5

The main idea has been used in the proof of Kanda [14, Theorem 2]. Denote by  $\tilde{\psi}$  the Lévy-Khintchine exponent of  $\tilde{X}$ . Then, for any  $\lambda > 0$ , we have

$$\operatorname{Re} \left( \frac{1}{\lambda + \psi(\xi)} \right) \leq \frac{1}{\lambda + \operatorname{Re} \psi(\xi)} = \frac{1}{\lambda + \frac{1}{2} \tilde{\psi}(\xi)} \leq 2 \operatorname{Re} \left( \frac{1}{\lambda + \tilde{\psi}(\xi)} \right).$$

By Kanda [13, Remark 2.1] or Hawkes [11, Theorem 3.3], we find that there exists a positive constant  $M$  such that for every  $\lambda > 0$  and every compact  $K$ ,

$$C^\lambda(K) \geq M\tilde{C}^\lambda(K), \quad (3.9)$$

where  $C^\lambda(K)$  (resp.  $\tilde{C}^\lambda(K)$ ) is  $\lambda$ -capacity of  $K$  relative to  $X$  (resp.  $\tilde{X}$ ). Since  $\tilde{X}$  is a symmetric Lévy process with bounded continuous transition densities, it satisfies (H), i.e., every semipolar set of  $\tilde{X}$  is a polar set of  $\tilde{X}$ . By Kanda [14, Theorem 1], we get

$$\lim_{\lambda \uparrow \infty} \tilde{C}^\lambda(K) = \infty \quad (3.10)$$

for every non-polar compact set  $K$  of  $\tilde{X}$ . (We remark that, more generally, (H) implies (3.10) under the weaker condition that  $\tilde{X}$  has resolvent densities, see Gettoor [7, Theorem (11.21)].) By the assumption, we find that every non-polar compact set  $K$  of  $X$  is a non-polar compact set of  $\tilde{X}$ . Thus, by (3.9) and (3.10), we get

$$\lim_{\lambda \uparrow \infty} C^\lambda(K) = \infty$$

for every non-polar compact set  $K$  of  $X$ . Then, by Kanda [14, Theorem 1] again, we obtain that every semipolar set of  $X$  is a polar set of  $X$ .  $\square$

### 3.5 Proof of Proposition 1.6

Suppose that  $d > 0$ . Then  $X$  is strictly increasing, which together with the right continuity of sample paths implies that singletons are thin and thus semipolar. By Kesten [15] or Bretagnolle [3], we know that  $X$  hits points with positive probability. Hence (H) cannot hold. Therefore we must have  $d = 0$ .  $\square$

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